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The effects of interference and viscosity in the Kelvin ship-wave problem

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A comparison is made between the effects of spreading the source (thus causing interference) and of viscosity in the Kelvin ship-wave problem. Two simple pressure distributions are considered and in both it is found that spreading, like viscosity, introduces a multiplicative damping factor into the inviscid pressure-point solution. In one case it is this factor, rather than that of viscosity, which dominates the decay of the wave profile, while, in the other, the effects alternate in importance as one travels along any particular wave crest.

1. Introduction

The classical solution to the Kelvin ship-wave problem has a singularity at the origin of disturbance. Ursell (1960) said that this would be eliminated if the pressure-point distribution were replaced by a spread source (thereby causing interference). An alternative method was given by Cumberbatch (1965), who, on introducing a little viscosity, found that the classical inviscid solution has to be multiplied by an exponential decay factor which takes the wave profile to zero at the origin. The two effects, one of spreading the surface pressure, the other of viscosity, are compared in this paper, and the results shown graphically in § 3. It is found, for the first of the two pressure distributions considered, that spreading introduces an exponential decay factor and that this is more important than the viscous effect for large Reynolds numbers at least. The second pressure distribution is one of finite extent, and the effects now alternate in importance as one travels along any particular wave crest. However, if the Froude number is greater than 0·627, it is the viscous factor which dominates the decay of the transverse wave system at large distances from the ship.

Pressure distributions are chosen which are easy to manipulate and which introduce a simple length scale. This length plays a significant role in the non-dimensionalization and is basic in the definition of the Reynolds number R_e for the problem. The work of Cumberbatch had no such natural 'ship' length and he was forced to use an artificial one (see his equation (19)). This meant that his Reynolds number in many practical applications was too small, which, in turn, overemphasized the effect of viscosity.

In the analysis of § 2 a double integral is evaluated asymptotically. The method is a consistent one avoiding the difficulties mentioned at the end of § 2 in Cumberbatch's paper, and showing his numerical results to be true to a high degree of accuracy.

2. Equations

Cumberbatch's formulation of the problem is taken as a starting-point. His delta-function normal stress is replaced in the first instance by a spread source of the form

$$G = -(P/\pi)\exp\{-(x^2+y^2)/a^2\}, \tag{1}$$

where P is constant and a is the fundamental length scale. The non-dimensionalization is redefined as

$$\bar{x} = x/a, \quad \bar{y} = y/a, \quad \bar{r} = r/a, \quad \bar{\alpha} = \alpha a f^2, \quad \bar{\beta} = \beta a f^2, \quad \bar{R} = R_e f^2 = Uaf^2/\nu, \tag{2}$$

where f is the Froude number, $U/(ag)^{\frac{1}{2}}$. One finds on dropping bars that the dimensional wave profile is now

$$\eta = -P(4\pi^2 f^4 \rho g)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i(\alpha x+\beta y)/f^2\}\exp\{-(\alpha^2+\beta^2)/4f^4\} \times D^{-1}(\alpha,\beta,R)\,d\alpha\, d\beta, \tag{3}$$

where D is given by Cumberbatch's equation (22).

It is convenient at this stage to leave Cumberbatch's analysis and follow the work of Crapper (1964) on the inviscid pressure-point problem. With a rotation of axes, (3) becomes

$$\eta = -P(4\pi^2 f^4 \rho g)^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i\bar{\alpha}r/f^2\}\exp\{-(\bar{\alpha}^2+\bar{\beta}^2)/4f^4\}D^{-1}\,d\bar{\alpha}\,d\bar{\beta}, \tag{4}$$

where

$$\alpha = \bar{\alpha}\cos\theta - \bar{\beta}\sin\theta, \quad \beta = \bar{\alpha}\sin\theta + \bar{\beta}\cos\theta; \quad x = r\cos\theta, \quad y = r\sin\theta. \tag{5}$$

By a residue calculation

$$\eta \sim -P(4\pi^2 f^4 \rho g)^{-1} 2\pi i \Sigma \int_{-\infty}^{\infty}\exp\{i\bar{\alpha}r/f^2\}\exp\{-(\bar{\alpha}^2+\bar{\beta}^2)/4f^4\}\left[\frac{\partial D}{\partial \bar{\alpha}}\right]^{-1} d\bar{\beta}, \tag{6}$$

for large r, the summation being over all values of $\bar{\alpha}$ for which $D = 0$. The determination of these $\bar{\alpha}$ is not easy. They can only be found approximately. Returning to Cumberbatch, and expressing the solution to $D(\bar{\alpha}) = 0$ as a power-series expansion in inverse roots of his Reynolds number R, one finds such $\bar{\alpha}$ have the form

$$\bar{\alpha} = A_1 + C_1 R^{-1} + O(R^{-\frac{3}{2}}), \tag{7}$$

where

$$0 = 1 - (A_1\cos\theta - \bar{\beta}\sin\theta)^2 (A_1^2+\bar{\beta}^2)^{-\frac{1}{2}}, \tag{8}$$

and

$$C_1 = \frac{-4i(A_1^2+\bar{\beta}^2)^2}{A_1(A_1\cos\theta - \bar{\beta}\sin\theta) - 2\cos\theta(A_1^2+\bar{\beta}^2)}. \tag{9}$$

By the transformation

$$\zeta = A_1\cos\theta - \bar{\beta}\sin\theta, \tag{10}$$

one can find expressions for A_1 and C_1 as functions of θ and ζ. Bearing in mind that, if R tended to infinity and the source were concentrated to a delta-function, Crapper's problem would be realized, it can be shown that A_1 must be real. If from now on terms $O(R^{-1})$ are neglected unless they appear in the form r/R, which is in its turn considered much smaller than r, then (6) can be evaluated by the method of stationary phase giving

$$\eta \sim -P(4\pi^2 f^4 \rho g)^{-1} 2\pi i \sum \exp\{i\bar{\alpha} r/f^2\} \exp\{-(\bar{\alpha}^2+\bar{\beta}^2)/4f^4\} \left[\frac{\partial D}{\partial \bar{\alpha}}\right]^{-1} \left|\frac{d\bar{\beta}}{d\zeta}\right| \left(\frac{2\pi}{Mr}\right)^{\frac{1}{2}} \times \exp\{\tfrac{1}{4}\pi i \operatorname{sgn} M\}, \quad (11)$$

where $$M = d^2\bar{\alpha}/d\zeta^2, \quad (12)$$

and $$\zeta^2 = \frac{1}{8\tan^2\theta}\{4\tan^2\theta + 1 \pm (1-8\tan^2\theta)^{\frac{1}{2}}\}. \quad (13)$$

This evaluation is only valid because $d\bar{\beta}/d\zeta \neq 0$ when ζ is given by (13), and θ is taken such that $|\theta| < \theta_c$ (see Crapper, §3). Carrying out the algebra in (11), one finds

$$\eta \sim aKf^{-2}[\exp\{-\tfrac{1}{4}\zeta^4\}\exp\{-r\cos\theta B/R_e\}]^{f^{-4}}, \quad (14)$$

where $$B = 4\zeta^6/(2\zeta^2-1), \quad (15)$$

and K is Crapper's inviscid solution for a pressure point.

If the infinite distribution of pressure given by (1) is replaced by a finite distribution of the form $-PH(a-r)$, where $H(\phi)$ is the Heaviside unit step function defined by

$$\left.\begin{aligned} H(\phi) &= 0 \quad \text{if} \quad \phi < 0, \\ &= 1 \quad \text{if} \quad \phi > 0, \end{aligned}\right\} \quad (16)$$

then the interference term, $\exp\{-(\alpha^2+\beta^2)/4f^4\}$, in (3) must be replaced by

$$\frac{2\pi J_1\{(\alpha^2+\beta^2)^{\frac{1}{2}}/f^2\}}{(\alpha^2+\beta^2)^{\frac{1}{2}}/f^2}$$

and (14) becomes $$\eta \sim aK2\pi\zeta^{-2}J_1(\zeta^2/f^2)\exp\left\{-\frac{r\cos\theta B}{f^4 R_e}\right\}. \quad (17)$$

3. Results

The variation of amplitude along a particular wave crest can be calculated from (14) by keeping $\alpha x + \beta y$ constant, say r_0 (see Crapper, §3). There is a phase shift of $\frac{1}{2}\pi$ on the line $\theta = \theta_c$ but this is ignored for the sake of clarity. The variation of ζ^4, the spreading coefficient in the first case, and of $r\cos\theta B/4r_0$, the viscous coefficient, are given in figure 1. They are plotted against variation in θ as one travels along a crest from $\theta = 0$ on the transverse wave to $\theta = \theta_c$ and then back again on the diverging wave (which goes to the origin). It is clear that the spreading coefficient is faster growing. In fact it is not difficult to prove that

$$4r_0\zeta^4/r\cos\theta B \to \infty$$

as $\theta \to 0$ on the diverging wave. Because of this the wave profile tends to zero at the origin ($K \to \infty$ non-exponentially).

Points at which the viscous and spreading factors in (14) are equal can be shown to lie on a circle. This circle, which touches the lines $\theta = \pm\theta_c$ and has non-dimensional radius $\frac{1}{32}R_e$, is illustrated in figure 2. The decay of waves in the first region (nearest the origin) is chiefly due to spreading, while viscosity is the more important effect in the third region. In between, spreading dominates the decay of the diverging wave, and viscosity that of the transverse wave. Since K is $O(r^{-\frac{1}{2}})$ and R_e is very large, all waves of any practical interest will be in the first region and hence dominated by the effect of spreading.

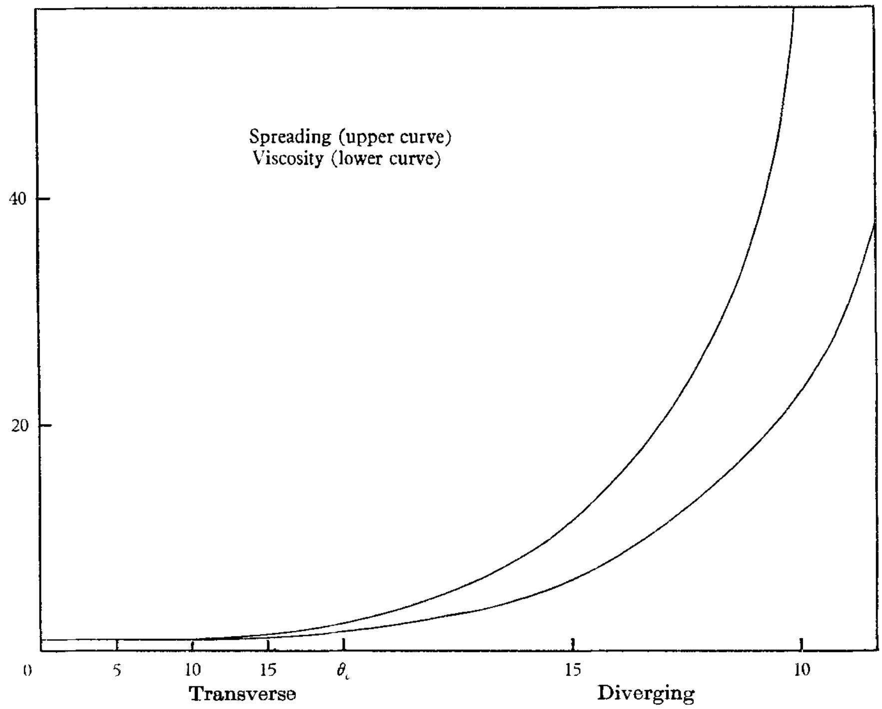

FIGURE 1

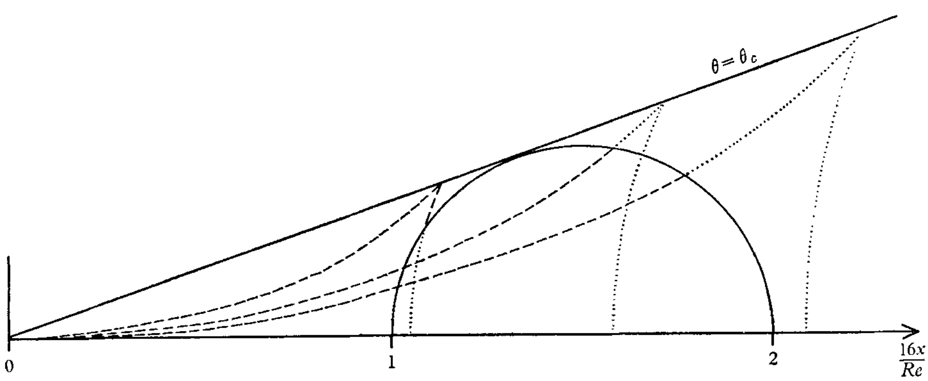

FIGURE 2. The wave crests shown are just three of many which might have been drawn. Their phase shift on the line $\theta = \theta_c$ has been ignored for the sake of clarity. The parts of the wave crests represented as dotted lines are dominated by viscosity, and those as dashed lines by spreading (interference).

The second pressure distribution is now considered. Since ζ^2, which is unity for $\theta = 0$ on the transverse wave, increases to 1·5 at $\theta = \theta_c$ and then tends to infinity as $\theta \to 0$ on the diverging wave, it is clear that the zeros of the oscillatory function $J_1(\zeta^2/f^2)$ in (17) lie on two pencils of lines within the wedge $|\theta| < \theta_c$. One set, finite in number, gives the position of the zeros on the transverse wave system as a function of the Froude number f, while the other set, which has a limit point at $\theta = 0$, gives the position of zeros on the diverging wave. The first set is the null set if $f > 0{\cdot}627$ and the zeros of $J_1(\zeta^2/f^2)$ affect only the diverging wave pattern. However, there exists an infinite set of discrete Froude numbers which are less than 0·627 for which $J_1(\zeta^2/f^2)$ is zero at $\zeta^2 = 1{\cdot}5$. These are Froude numbers for which the large amplitudes at $|\theta| = \theta_c$ (see Crapper's equation (38)) can be eliminated and for which one might expect a reduction in wave drag. In fact $f = 0{\cdot}627$ corresponds very closely to the maximum Froude number, as calculated by Barratt (1965), for which the wave resistance of a circular hovercraft has a local minimum.

While it is obvious that viscous decay dominates the transverse wave system at great distance from the ship (large r in (17)) provided $f > 0{\cdot}627$, it is not clear as one travels in along a diverging wave whether this effect, rather than that of interference due to spreading, is more important. Although the viscous term tends to zero exponentially, the interference term is periodically zero. Nevertheless, given any r, f and R, one can calculate the sequence $\theta_1, \theta_2, \theta_3, \ldots$, where it is possible to say viscosity is more important in the range $\theta_n > \theta > \theta_{n+1}$ for n even, say, and interference more important for n odd. There are two such sequences; the one for the transverse waves is finite, while that for the diverging waves has a limit point at $\theta = 0$.

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